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The motion generated by a body moving along the axis of a uniformly rotating fluid

By W. G. PRITCHARD
Department of Mathematics,
The University of Manchester Institute of Science and Technology

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Experiments have been made to investigate the motion generated by a body moving along the axis of a uniformly rotating fluid.

Part of the investigation concerns the motion generated in a cylinder whose radial dimensions are much greater than those of the body. Measurements have been made of the velocities of particles on the axis of rotation both ahead of and behind the body, and the results indicate that there is a significant axial motion generated by the body over a wide range of Rossby numbers. A measurement of the instantaneous velocity profile ahead of the body, determined as a function of the radius, agrees fairly well with a low Rossby number calculation of the flow due to a circular disk (Morgan 1951). In addition, the forward influence of the body has been measured as a function of the Rossby number and from these results it is suggested that the body has a finite influence far upstream at *all* Rossby numbers and that the blocking phenomenon first reported by Taylor (1922) probably occurs for all values of the Rossby number ($U/\Omega a$) less than a critical value which is about 0·7.

Experiments have also been made in a long cylindrical tube which acts as a wave guide. At large distances from the body the separate effects of the various modes can be observed and hence it is possible to measure the flow corresponding to an individual wave-number: these measurements show that, as a result of the Doppler effect, the motion a large distance ahead of the body is different from that far behind (see Lighthill 1967). Moreover, the experiments indicate that no disturbances propagate ahead of the body when its velocity exceeds the maximum group velocity in the fluid, but that disturbances trail behind the body when its velocity is far in excess of the maximum group velocity. Measurements of the maximum group velocity are in good agreement with the theoretical value.

1. Introduction

The experiments described in this paper were made in order to obtain some detailed measurements of the flow generated by a body moving along the axis of a uniformly rotating fluid. The motions have been investigated over a wide range of values of the Rossby number.

Observations of the flow due to an obstacle of radius a moving with a speed U along the axis of rotation of a fluid which has angular velocity Ω suggest that a column of fluid is pushed ahead of the moving body when $U/\Omega a$ is less than about 0·2 or 0·3 (cf. Taylor 1922; Long 1953). Taylor's (1922) experiment used a very light sphere painted in stripes to make visible any rotation. When the sphere has no axial motion it naturally rotates with the fluid, but if it is allowed to move along the axis with a velocity U such that $U/\Omega a$ is greater than about $1/\pi$, Taylor found that the sphere ceases to rotate with the fluid. If, on the other hand, the sphere moves slower than this it continues to rotate with the fluid and an investigation of the flow with coloured water indicated that a column of fluid, of the same diameter as the sphere, is pushed along in front of the sphere. This blocking effect was also observed by Long (1953) using a streamlined body in a tube of radius $4a$, but he estimated that fluid is pushed ahead of the body only when $U/\Omega a$ is less than about 0·23. In addition Long observed wavelike motions extending a great distance behind the body.†

An investigation of the blocking phenomenon has also been made by Benjamin & Barnard (1964) in the course of their study of the motion of a cavity progressing along the axis of a long column of rotating liquid. They observed the flow ahead of the cavity from the distortion of sheets of tellurium dye that spanned the tube and found substantial forward influence of the cavity at values of $U/\Omega b \approx 0{\cdot}38$ (where b is the radius of the tube), but the nature of the experiment was such that this parameter could not be varied. From their measurements Benjamin & Barnard made a rough estimate of $0{\cdot}48\Omega b$ for the group velocity.

Recently, Maxworthy (1968) has investigated the motions in a short cylinder when $U/\Omega a < 0{\cdot}1$. An important feature of his experiments is the examination of the boundary layers on the sphere: the fluid ahead of the sphere and within the Taylor column rotates more slowly than the obstacle; fluid is sucked smoothly into the Ekman layer on the front of the sphere, passes to the rear of the obstacle and is ejected from the layer behind the sphere. Maxworthy also found that the flow from the rear Ekman layer is not well behaved and has the appearance of a mild vortex breakdown.

The theoretical work of Morgan (1951), Stewartson (1952) and Bretherton (1967) indicates that a body, started moving along the axis of rotation at a uniform velocity such that $U/\Omega a \ll 1$, ultimately has a stagnant column of fluid extending far ahead of and behind itself. Bretherton's analysis, which was made for a strictly two-dimensional flow, investigates the motions at large but finite times; the solution covers the regions far from the body where the disturbance has only just arrived, the region near the edge of the Taylor column, and also the region near the body: in all cases the development of the flow may be attributed to inertial waves.

Studies of the motions at finite values of the Rossby number have been made by Stewartson (1958, 1968) and Lighthill (1967). Stewartson (1958) attempted

† Long found in some of his experiments that the wavelength of the motions was slightly less than the theoretical value; he attributed this to the low value of the ratio of the length to the diameter of the cylinder used for the experiments. Recent measurements (Pritchard 1968) made in cylinders of larger length-to-diameter ratios confirm these results: the motions whose wavelength is less than about 0·4 times the length of the cylinder are in general agreement with Long's theory, whereas the longer wavelength motions have a smaller wavelength than the theoretical value.

to give a theoretical explanation of Taylor's (1922) experiments: his results suggest that when $U/\Omega a = 0{\cdot}348$ no axisymmetric steadily translating solution exists for an inviscid fluid in which the flow far upstream is merely rigid-body rotation;† moreover, his calculations indicate that, at higher Rossby numbers, the drag on the sphere has unusually large values in the absence of upstream influence. Thus Stewartson proposes that ultimately, for all Rossby numbers less than a critical value which is at least 0·348, the flow far upstream differs substantially from that of uniform rotation. Lighthill's (1967) analysis shows that, when a body moves along the axis of rotation, those waves with a group velocity greater than the velocity of the body propagate indefinitely far ahead of the body without attenuation. Hence, if all wave modes are excited, the body should have a forward influence at all Rossby numbers. Following these considerations, Greenspan (1968) has conjectured that a column of liquid is trapped in front of the body (the blocking phenomenon)‡ at all Rossby numbers less than about 0·7. Greenspan bases his suggestion on the solution for the motion generated by a disk that moves slowly along the axis of rotation after an impulsive start: in the flow ahead of the disk there is a 'stagnation point' behind which exists a reverse cellular flow. This stagnation point advances with a velocity $0{\cdot}675\Omega a$ and Greenspan suggests that, as a rough estimate, when the disk moves faster than this the blocking effect will not occur. The present experiments indicate that there is a forward influence at all Rossby numbers, although the Taylor column is very feeble at large values of the Rossby number, and that the blocking effect probably occurs for all Rossby numbers less than some critical value which is about 0·7. Stewartson's (1968) analysis supposes the motions to be governed by Oseen's equations from which he deduces that, even at very large Rossby numbers, there is a feeble wake both upstream and downstream of the body, but the analysis predicts the incorrect sign for the swirl velocity in the upstream wake; at small Rossby numbers he finds that strong wakes are set up, the upstream wake being almost like a 'plug' flow. In very recent correspondence, Dr T. B. Benjamin has indicated that he has found a proof (using the full non-linear equations of unsteady motion) that in inviscid rotating fluids, bodies moving axially have a finite forward influence at *all* Rossby numbers (except when $U/\Omega b > 0{\cdot}522$ for flow in a tube of radius b; $0{\cdot}522\Omega b$ is the critical speed for infinitesimal waves).

An additional feature of Lighthill's (1967) analysis is that, as a result of the Doppler effect, a given wave mode is more powerfully excited ahead of the body

† This singularity is a spurious consequence of truncation (Miles 1968), but Stewartson's (1969) more recent computations still indicate unusually large drag coefficients in the absence of an upstream influence. On the other hand, Miles (1969*b*) suggests that, for values of $U/\Omega a$ greater than about 0·9, the difference between Stewartson's calculations and observation may be a consequence of viscous separation effects, and that at lower Rossby numbers the solution implied by Long's hypothesis may be invalid because of local reversals of the flow leading to instabilities.

‡ The term 'blocking' refers to the situation in which a quantity of dyed fluid placed in front of the body is pushed ahead of it for all time. On the other hand a body may have a finite 'foward influence' extending far upstream but give rise to a flow in which dyed fluid placed in front of the body does *not* remain there for all time.

than behind it. Measurements on a single wave mode, made in the course of the present experiments, are in agreement with this prediction.

Two sets of experiments are described in the paper. One set concerns the motion in a cylinder of large radial extent; the experiments were made in such a fashion that the ends of the container had only a very small influence on the flow. The other experiments were made in a long cylindrical tube: in these experiments attention was, necessarily, focused on the way in which the normal modes propagate along the tube. Since great care is needed in making measurements in rotating fluids, some of the more technical aspects of the work are discussed in detail (for example, the measurements made on the normal modes) in order to give confidence in the experimental technique and results.

2. Experimental apparatus and procedures

Flow measurement techniques

The measurements of the fluid motions were made from observations of dye traces produced by the pH-indicator technique described by Baker (1966). This flow-visualization technique employs a pair of thin platinum wires (0·001 in. diameter for the present experiments) immersed in a solution containing the indicator Thymol Blue. The solution initially has its pH very near that of the indicator change and when a small voltage is applied between the wires the change of pH in the immediate vicinity of the cathode causes the indicator to turn blue, thereby serving to mark a thin cylindrical region of fluid around the wire. The movements of these dye traces were recorded on 16 mm ciné film, and measurements of the relative positions of a trace were made directly from the film using a projection microscope. The microscope used for the measurements has a movable table, on which the film is placed, and which may be positioned to 0·0001 in. An image of the film was projected onto a screen and magnified by a factor of 10, 20 or 50, so that distances could be measured from the projected image or found as lengths on the film. In many of the experiments this method has detected a displacement of a fluid particle of less than 0·01 in.

Apparatus to investigate the motions in a fluid of large radial extent

A set of measurements was made in a Perspex cylinder (12 in. in diameter and 24 in. in length) in which disturbances were generated by a sphere ($1\frac{1}{4}$ in. in diameter) which was towed along the axis at a steady speed. The fluid motions were recorded on ciné film by photographing the dye traces generated at two platinum wires that stretched diametrally across the cylinder: one wire was located about 10 in. from the bottom of the container and the second wire about 6 in. from the bottom. The arrangement is shown schematically in figure 3, to follow. An attachment was put on the cylinder to provide a plane viewing surface in order to reduce to negligible proportions the optical distortion introduced by the curved walls of the container.

The complete apparatus was mounted on a turntable which was rotated at a uniform angular velocity: this arrangement meant that the sphere rotated with the same angular velocity as the container, even when it was being towed. Taylor (1922) showed experimentally that a sphere with a very small moment of inertia stops rotating if moved quickly along the axis of a rotating fluid, therefore the imposed rotation of the sphere may, in the experiments described below, affect the dynamics. On the other hand, it should be noted that the flow in the boundary layers on the sphere (cf. Maxworthy 1968) consists of a suction into the boundary layer on the forward half of the sphere and hence it is anticipated that the 'Taylor column' ahead of the sphere is not as powerfully excited as it would be if the fluid were inviscid; moreover, Stewartson (1968) has suggested that, at high Rossby numbers, there are only small differences in the forward wake between the cases in which the body rotates and in which it does not rotate.

For all these experiments the Ekman number ($\nu/\Omega a^2$) was less than 10^{-3} and the Reynolds number ($2aU/\nu$) was larger than 500, where ν is the kinematic viscosity of the fluid.

Apparatus to investigate the motions in a long cylindrical tube

The apparatus used to investigate the motions due to a body moving along the axis of a long cylindrical tube is shown schematically in figure 1. The Perspex cylinder, which is 2 in. in diameter and 6 ft in length, is mounted on a roller bearing and is held vertical by three small roller bearings near the top. In the experiments the tube was rotated at a steady angular velocity (usually about 2π rad/sec) by means of a belt drive near the bottom. The long cylindrical body near the top of the tube was used to generate the disturbances to the primary flow.

About half the body was immersed in the liquid, and the tube and body were rotated for about 10 min to enable the liquid to spin up to a state of rigid-body rotation.† The body was then displaced along the axis at a steady speed and the resultant disturbance observed, by use of the Thymol Blue technique, as it progressed along the tube. The positions of the platinum wires, at which the dye lines were generated, are shown in figure 1; the wire that is parallel to the axis of rotation was used to produce a cylinder of dyed fluid by generating the 'dye' before the liquid had reached rigid-body rotation: this cylinder of dyed fluid was used to give information about the radial displacements associated with the disturbances. The movements of the dye lines were recorded on ciné film and, by using the arrangement of mirrors shown in figure 1, two sections of the tube were photographed on each frame of the film.

† Theoretical calculations (Pritchard 1968) show that, for water in an infinitely long 2 in. diameter tube, the flow differs from rigid-body rotation by less than 0·5 % after 5 min; the same fluid in a tube of finite length spins up more rapidly than this.

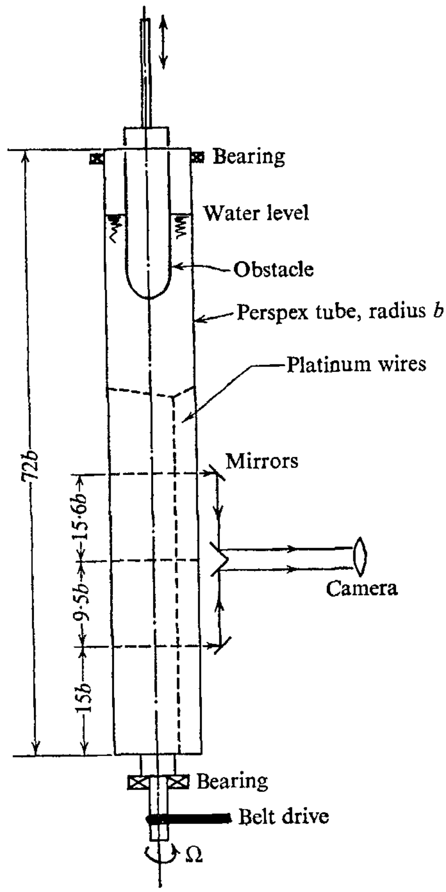

FIGURE 1. Schematic representation of the apparatus used to investigate the motions in a long cylindrical tube.

3. Motions in a fluid of large radial extent

The form of the disturbances

The motion ahead of and behind the sphere has been determined from the distortion of a dye line normal to the axis: an example of the way the line distorts when the sphere is moved towards it is shown in figure 2, plate 1. From a series of ciné photographs of this kind the displacement of a particle on the axis is easily determined (because of the axial symmetry of the motions) as a function of the time, and the velocity of the particle is computed from a smooth representation of this data.† However, restrictions on the length of the vessel have only allowed observations of time-dependent motions. Maxworthy (1968) has shown, for steady-state motions at low Rossby numbers, that the ends of the container have a marked effect on the flow field, but in the present time-dependent experiments *a posteriori* arguments (based on the measurements of the forward

† Details of the particle displacements are not given in this paper, but an example of similar results is included in the discussion on the motions in a long tube.

disturbances as a function of the Rossby number, and assuming exact reflexion of the waves) indicate that the observed flows are not likely to be influenced by more than 5 %, in the worst case, by the presence of the ends.†

Results of some of these experiments are shown in figure 3 (*a*). The curve labelled A shows the motion ahead of the sphere which had been moved towards the observation point at $t = 0$; initially the sphere was 20·3 radii from the nearer platinum wire. The Rossby number $(U/\Omega a)$ for experiment A was 0·68, and there is a considerable forward influence due to the sphere: the observed dye particle has attained half the velocity of the sphere when it is 6 radii away, and is accelerated to $0{\cdot}65U$ when the sphere is still 4·5 radii from it.

A similar experiment is presented in curve B: in this case the Rossby number is 0·83 and the motions are shown at two positions along the axis. At $t = 0$ the sphere was 14·6 radii from wire 1; its motion was stopped at $\Omega t/2\pi = 2{\cdot}38$. Again, substantial disturbances propagate ahead of the sphere, but they are not as strongly excited as in experiment A. This experiment was repeated at the same Rossby number, but instead of stopping the sphere near the wire its motion was arrested at $n = 2{\cdot}3$ and then reversed: the resulting flow is shown in curve C. The initial portions of curves B and C are almost identical, but shortly after the motion of the sphere is reversed its effects are noticeable at both dye traces and eventually there is a return flow.‡ The different forms of the velocity curves at the two observation points clearly demonstrate the dispersive effects of the medium.

Curve D shows the motion behind the sphere when it is moved (at $t = 0$) away from the observation point at a Rossby number of 0·98. Initially the sphere was 8·3 radii from wire no. 2. The results show that a considerable disturbance occurs behind the sphere.

A comparison of the results u_2 (experiment B) with theoretical estimates of the forward influence is shown in figure 3 (*b*). One of these theoretical estimates is the flow generated by a dipole moving along the axis of rotation (Miles 1969*a*) and the other estimate has been made from the low Rossby number motions generated by a disk moving along the axis of rotation (see Greenspan 1968, and equation (2) to follow). The theory of Miles is based on the hypothesis of Long (1953) that the disturbances far upstream are arbitrarily small, whereas the other estimate attributes the motions entirely to inertial waves generated by the body and hence may lead to finite disturbances far upstream. The latter theory is not strictly applicable to experiments made at Rossby numbers of $O(1)$, but if the motions at the time and point of observation have been influenced only by wave modes whose group velocity is much greater than the velocity of the body,

† It has been shown by Moore & Saffman (1968) that the Taylor column in a short cylinder is different from the column formed in the unbounded case. For a cylinder of depth h they showed that if $h/a \ll E^{-\frac{1}{2}}$, where $E = \nu/a^2\Omega$ is the Ekman number, the swirl is $O(UE^{-\frac{1}{2}})$ in contrast to the swirl $O(U)$ in the unbounded $(h/a \gg E^{-1})$ case. However, this flow needs a time $\surd(h^2/\Omega\nu)$ to be established and for shorter times the effect of the ends is to reflect the waves. In the present experiments the spin-up time is about 3–5 min which is to be compared with a time of about 10 sec for the motion of the body.

‡ Not shown in these results, but evident from the measurements of the displacements, is an additional, fairly weak, disturbance apparently generated by the discontinuity in the motion of the sphere. It was observed on a number of occasions and its propagation speed, in all cases, was about $1{\cdot}6\Omega a$.

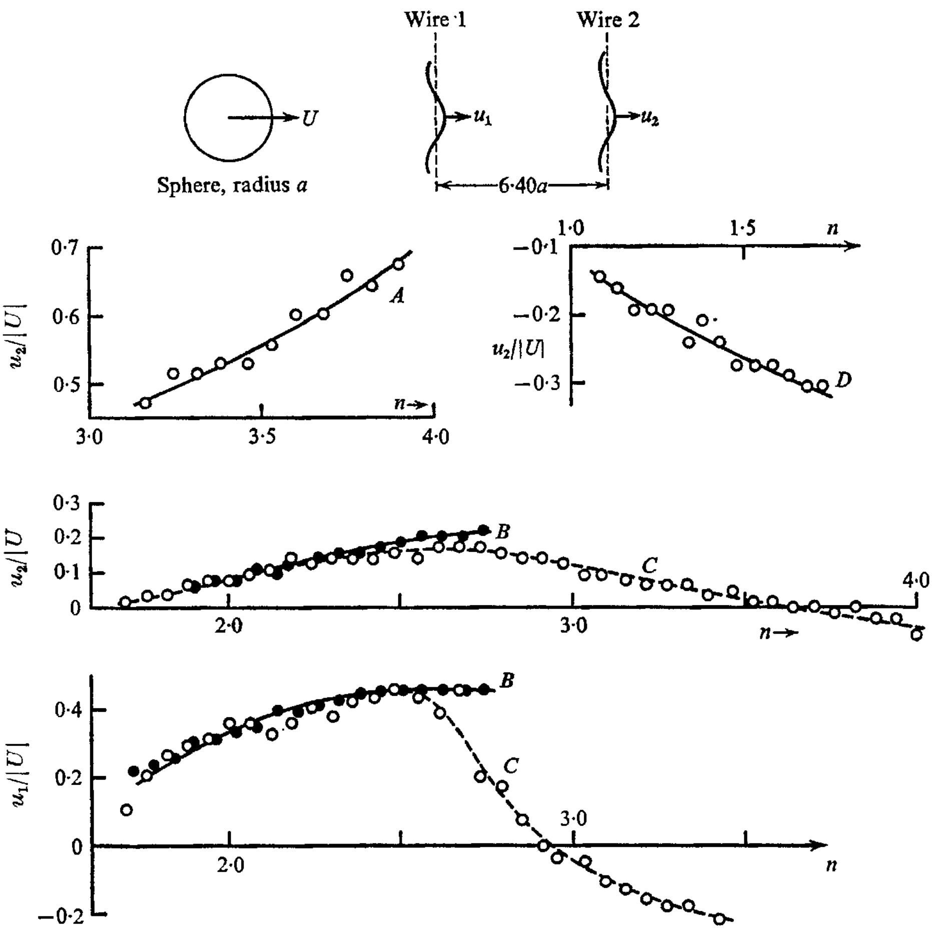

FIGURE 3(*a*). The motion due to a sphere moving along the axis of a uniformly rotating fluid of large radial extent. $n(=\Omega t/2\pi)$ is the number of the tube periods after the beginning of the motion. A, sphere moved towards wires, $U/\Omega a = 0{\cdot}68$; B, sphere moved towards wires, $U/\Omega a = 0{\cdot}83$; C, sphere moved towards wires, then away, $U/\Omega a = 0{\cdot}83$; D, sphere moved away from wires, $U/\Omega a = 0{\cdot}98$.

the theory should be a reasonable approximation to the experiments. Thus, to compute the low Rossby number estimate of figure 3(*b*) we assume that the body moves a negligible distance along the axis during the course of the experiment, so that the distance from the body is given by its value at the beginning of the motion. We see that the agreement of both theories with the experimental results is fairly good, but it is stressed that, in view of the approximations involved, no particular significance should be placed on the slightly better representation of the data given by the estimate made from the low Rossby number theory.

The instantaneous velocity profile

A measurement has been made of the instantaneous velocity profile ahead of the sphere, determined as a function of the radius. The measurement was made

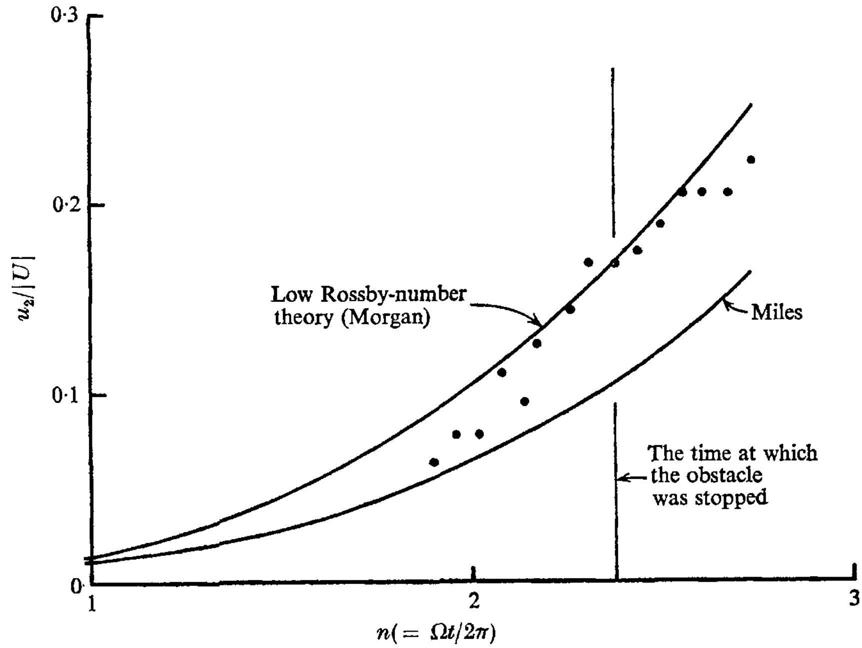

FIGURE 3(*b*). Comparison of $u_2/|U|$ of figure 3(*a*) (experiment *B*) with theory.

from the distortion of a dye trace of the kind shown in figure 2, plate 1, with the assumption that the radial velocities are negligible compared to the axial velocities, so that the particle paths are nearly parallel to the axis.

Morgan (1951) has calculated the development of the flow field due to a disk suddenly started moving along the axis. Although this calculation is based on the assumption of very small Rossby numbers, it may (as suggested above) give reasonable estimates of the motions observed in experiments made at high Rossby numbers, if we restrict our attention to that part of the flow field which is influenced only by wave modes whose group velocity is much greater than the velocity of the body. Thus we look at that part of the Taylor column which is influenced only by wave modes with large transverse wavelengths: in general, transverse wavelengths greater than $2\pi/(2\Omega at/x)$ affect the motion at the particular time and point under consideration.

An experimental curve of the instantaneous velocity profile is given in figure 4. This experiment was made at a Rossby number of 0·83 and the value of $2\Omega at/x$ is 1·8, assuming that x is determined from the position of the sphere at $t=0$. The theoretical curve is the velocity profile at $2\Omega at/x = 1\cdot8$ for the axisymmetric flow generated by a circular disk (cf. Greenspan 1968 and Morgan 1951). The agreement between the experimental results and the theoretical calculations is fairly good, especially in view of the fact that, at the instant of measurement, the centre of the sphere was only 4·5 radii from the dye trace, and that the experiment was made at a Rossby number of $O(1)$.

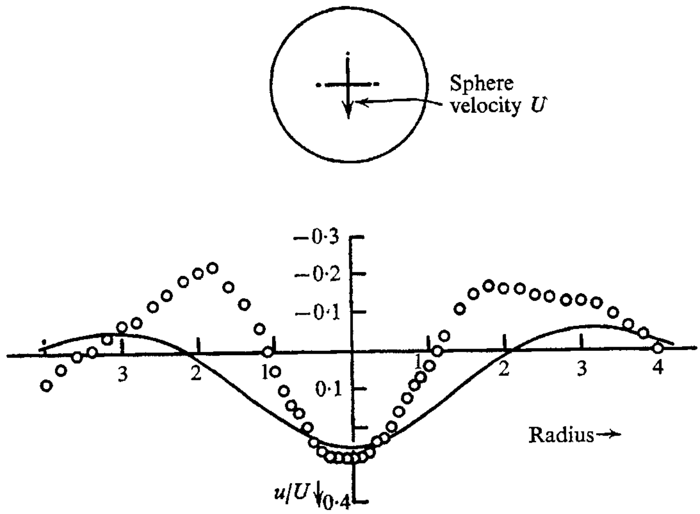

FIGURE 4. The instantaneous velocity profile ahead of a sphere moving along the axis of a uniformly rotating fluid of large radial extent. $t/x = 1{\cdot}8$. ———, theory (circular disk); ○, experiment.

The forward influence as a function of the Rossby number

An experiment has been made to determine the influence ahead of a body as a function of the Rossby number. The experiment is slightly complicated because we are forced to make the measurements before the flow has had time to reach its steady state. Thus, suppose the situation at time t after the beginning of the motion of the body is as shown in the sketch (figure 5); from dimensional considerations the velocity is determined by a relation of the form

$$u/U = f(U/\Omega a, x/a, \Omega t). \tag{1}$$

Hence we see that it is a relatively simple matter to measure u/U whilst keeping both x/a and Ωt constant, but allowing $U/\Omega a$ to vary. A measurement of u/U has been made at $x/a = 4$ and $\Omega t = 12{\cdot}5$; the results are given in figure 6. Also shown in figure 6 are two (fairly crude) theoretical estimates of the steady state results. These estimates are based on low Rossby number theories with the assumption that, at high Rossby numbers, the waves whose group velocity is less than the velocity of the body are eliminated from the Taylor column ahead of the body.

For example, if we consider the axisymmetric flow generated by a disk moving slowly along the axis we see that the velocity (u) of a particle on the axis is (cf. Greenspan 1968, equations 4.3.1 and 4.3.13)

$$\begin{aligned} u/U &= -\frac{2}{\pi}\int_0^{2\Omega at/x}\left(\cos k - \frac{\sin k}{k}\right)dk \\ &= \frac{2}{\pi}\{Si(2\Omega at/x) - \sin(2\Omega at/x)\}. \end{aligned} \tag{2}$$

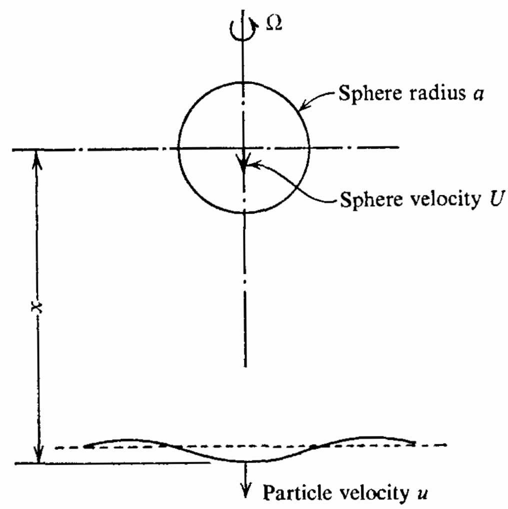

FIGURE 5. Schematic representation of the method used to determine the motion ahead of a sphere moving along the axis. ———, distorted dye line; -----, original line; —·—, sphere position at time t.

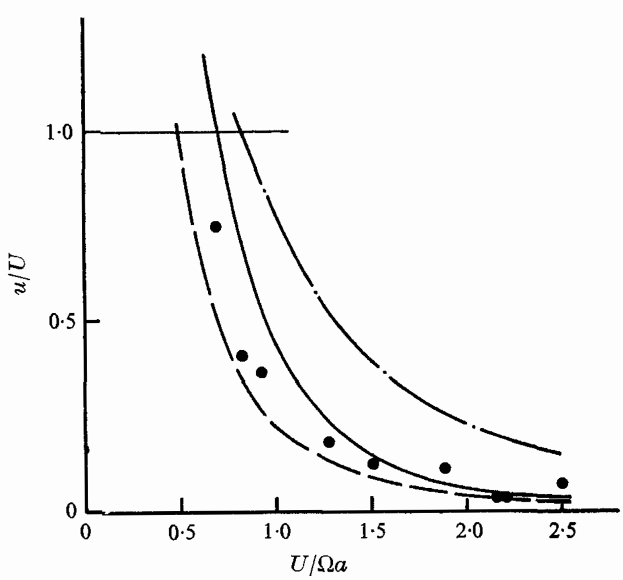

FIGURE 6. A measurement of the velocity ahead of a sphere as a function of the Rossby number. $x/a = 4$; $\Omega t = 12{\cdot}5$. Curves (a) and (b) are approximate theoretical estimates of this function. ———, (a) Morgan (circular disk); —·—, (b) Bretherton (two-dimensional theory); — — is equation (2) with x represented by its value at $t = 0$.

The group velocity of a mode with wave-number k is $2/k$, and hence the upstream motions after a time t and at a distance x are determined by those wave modes whose group velocity C_g exceeds $x/\Omega at$. For bodies moving with a velocity U of $O(1)$ it is anticipated that the upstream motions will be influenced only by those wave modes whose group velocity exceeds that of the body. Accordingly a 'rough' estimate of the steady state influence of the body is obtained by taking finite values for the upper limit $2\Omega a/U$ (where U now replaces x/t) of the integral (2), instead of the infinite value this limit has for the low Rossby number theory. With the integration limits taken thus, equation (2) represents one of the theoretical curves of figure 6; one of the others is the analogous result obtained from the two-dimensional flow generated by a cylinder (cf. Bretherton 1967, equation (6.8)). But it is stressed that these are only rough estimates since we are neglecting both the non-linearities and the Doppler effect, and we are assuming that the spatial combination of the wave modes is the same as that of the low Rossby number case. Yet, in spite of the approximate nature of this estimate, we see from figure 6 that the theoretical curve (*a*) gives a very good prediction of the experimental results determined at a finite time after starting the motion of the body. It is felt that this agreement suggests that the motion of the sphere generates the complete spectrum of wave modes and hence (see Lighthill 1967) the influence would extend indefinitely far upstream in an unbounded inviscid fluid. On the other hand, Miles (1969*b*) has shown it is possible to give a very good theoretical account of the results of figure 6 on the basis of a local disturbance (namely one that becomes arbitrarily small at large distances from the obstacle) generated by a dipole.

The third theoretical curve of figure 6 represents equation (2) as it stands, with x taken to be the distance of the sphere from the dye line at $t = 0$. We see that, as with the data of figures 3 and 4, this theoretical estimate represents the data fairly well.

4. Motions in a long cylindrical tube

Preliminary considerations

In view of Taylor's (1922) paper, which showed that a very light sphere moving slowly along the axis rotates with the same angular velocity as the fluid, some preliminary experiments were made to determine the importance of having the sphere rotate with the fluid. If the sphere moves along the axis but is not allowed to rotate, the fluid motions may be significantly influenced by the secondary flow generated by the Ekman layers on the body. For example, an experiment was made in which a 1 in. diameter sphere was held motionless in the long, 2 in. diameter tube, and the tube rotated at a steady speed of about 2π rad/sec: due to the presence of the sphere a strong secondary flow was set up in which the axial velocity was found to be $0{\cdot}23\Omega a$, where a is the radius of the sphere. Accordingly an experiment was made in which a body was moved slowly along the axis of the tube and the velocity of a particle on the axis measured at a position 45 tube radii ahead of the body. It was found that, when the body did not rotate, the velocity at this point was about 20 % greater than when it rotated. On the other hand, Stewartson's (1968) calculation indicates that, at high Rossby numbers, this effect is probably not important. In all the experiments described the body was made to rotate at the same angular velocity as the tube.

The disturbance due to a body moving along the axis

In figure 7, plate 2, is a sequence of photographs showing, at successive half-periods of rotation, the passage of a wave along the tube. The dye line indicating the motions originally stretched across a diameter of the tube and was coincident with the platinum wire near the top of the photograph. A length scale is indicated by the outer diameter of the tube: it is $2\frac{1}{4}$ in. The wave form in this experiment was generated by moving the body along the axis a distance of $1{\cdot}5b$ (where b is the radius of the tube) and then stopping it.

The velocity distribution in this wave has been determined by measuring the velocity of the particle on the axis, which is located from the symmetry of the dye line; some results are given in figure 8. The motions were recorded at two stations: one was 25 tube radii from the body and the second a further 16 radii along the tube.† The unbroken curve shows the displacement of a fluid particle on the axis when the body was moved towards it and the dashed curve represents the particle displacement when the body was moved in the opposite direction. (It is not apparent from figure 8 that these displacements are actually in opposite directions.) The particle velocities were derived from the (smoothed) results for the particle displacement. The results indicate that the motions ahead of the body are of a similar form to those behind, except that the maximum velocity within the two wave forms is slightly different. At the first observation point the particle moved along the axis of the tube a distance of about 1·2 radii, stopped for a short time, and then was accelerated into motion again: this second movement is due to the arrival of the second mode of the motion and is discussed in greater detail below.

The normal modes

A wave form progressing along a cylindrical tube and constituting only a small disturbance to the primary state of uniform rotation may be represented by an axial velocity distribution of the form (cf. Long 1953; Benjamin & Barnard 1964)

$$u = \sum_{n=1}^{\infty} \alpha_n J_0(k_n r) f_n(x,t), \tag{3}$$

where x is the axial co-ordinate and J_0 is the zero-order Bessel function of the first kind; f_n is some function of the axial position and of the time; the radial wave-number k_n is determined by

$$k_n b = j_{1,n}, \tag{4}$$

in which b denotes the radius of the tube and $j_{1\,n}$ is the nth positive zero of J_1.

† The amplitude of the wave form decreases as it progresses along the tube due to dispersion and to viscous effects in the boundary layers on the wall of the tube. Approximate calculations (Pritchard 1968) suggest that these two factors account for the change of amplitude of the wave form. This contention is supported by measurements of the amplitude change of a solitary wave (which does not disperse) produced under similar conditions: in this case the viscous effects alone give a good prediction of the amplitude change.

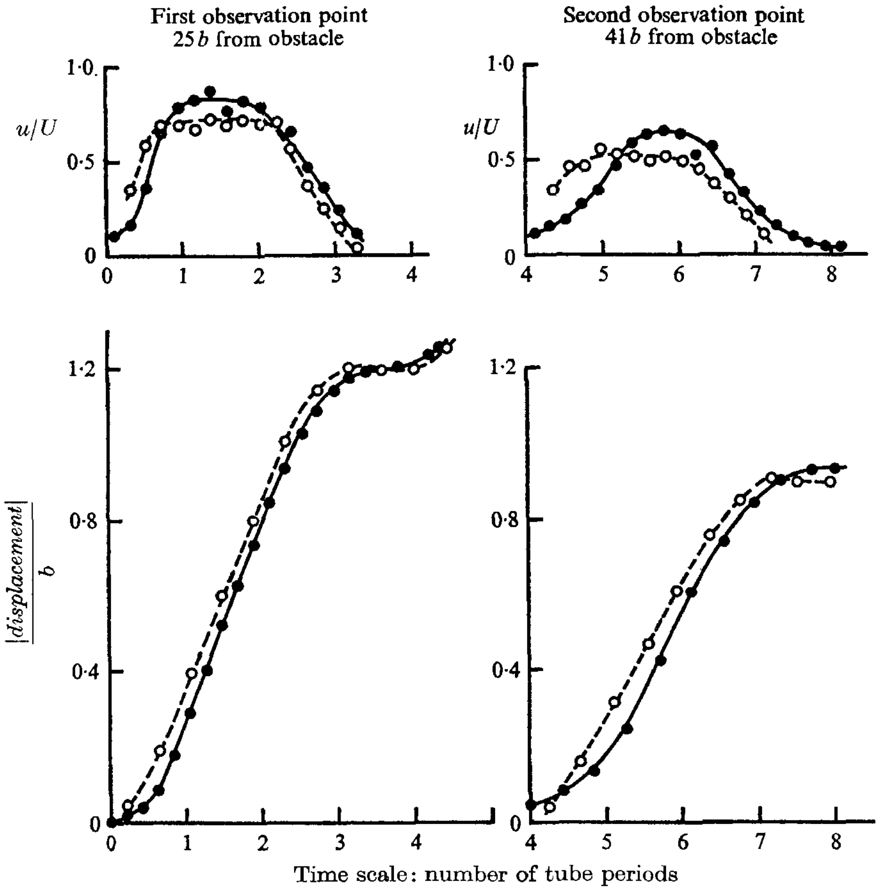

FIGURE 8. The motion due to a body moved along the axis of a long cylinder of rotating liquid a distance of $1\cdot5b$. $U/\Omega b \sim 0\cdot1$, $a/b = 0\cdot5$. ———, motions ahead of the body; – – – –, motions behind the body.

Hence, if at a given instant and position in the tube the velocity distribution is known, we can determine the normal-mode decomposition of the wave form. Thus, for example, suppose the axial velocity has a 'top-hat' distribution as shown in figure 9: the coefficients α_n are then easily evaluated by the usual method for orthogonal series, a Fourier–Bessel series in this case, and we find that

$$\alpha_n = \frac{2}{j_{1,n}(b^2/a^2-1)}\frac{J_1[(a/b)j_{1,n}]}{J_0^2(j_{1,n})}. \tag{5}$$

We anticipate that a body of radius a, moving slowly along the axis, generates a velocity distribution which is approximately of the form shown in figure 9: accordingly the amplitudes of the modes should be fairly well represented by (5). When a/b is greater than about $0\cdot55$ the sign of the second mode is negative and thus, on the axis, it has a velocity in the opposite direction to that of the fundamental mode.

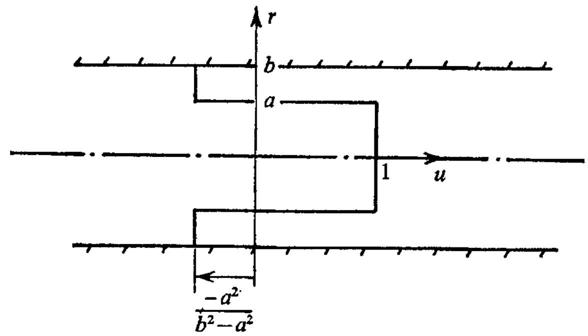

FIGURE 9. The axial velocity profile used for a normal-mode decomposition.

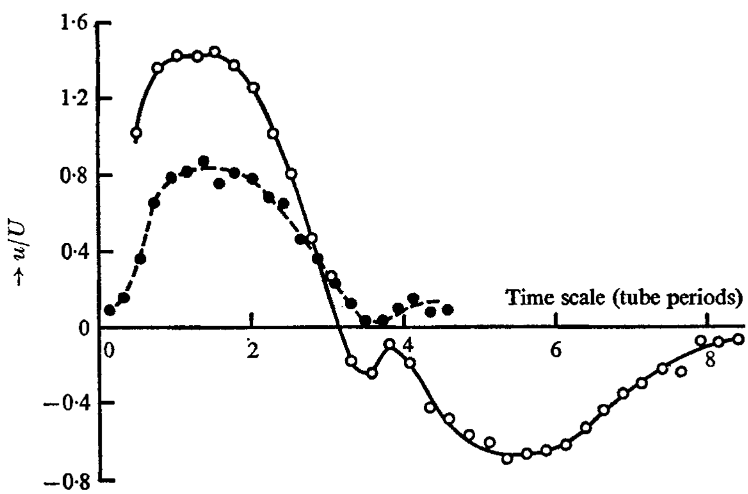

FIGURE 10. A comparison of the motions due to two bodies of different diameters which were moved along the axis. ----, (*a*) $a/b = 0{\cdot}50$, $U/\Omega b = 0{\cdot}100$, body moved a distance $15b$. ———, (*b*) $a/b = 0{\cdot}70$, $U/\Omega b = 0{\cdot}045$, body moved a distance $0{\cdot}8b$. Initially the body was $25b$ from this observation point.

The group velocity of each mode is directed along the axis and has a magnitude of $2\Omega b/j_{1,n}$ (Benjamin & Barnard 1964). Thus, if an obstacle is moved along the axis a short distance and stopped, the various modes will separate as the wave progresses along the tube. To demonstrate this effect an obstacle with $a/b = 0{\cdot}5$ was moved along the axis a distance of 1·5 tube radii and stopped; the resulting wave form was observed at a station 25 tube radii ahead of the body, and the results of this measurement are given in figure 10. When $a/b = 0{\cdot}50$ we see from (5) that $\alpha_1 = 1{\cdot}23$ and $\alpha_2 = 0{\cdot}27$; the velocity measured in the wave form was $0{\cdot}82U$ and the results indicate the arrival of the second mode. This experiment was repeated with a body for which $a/b = 0{\cdot}70$, so that $\alpha_1 = 1{\cdot}98$ and $\alpha_2 = -1{\cdot}36$. In this case the body was moved along the axis a distance of 0·8 tube radii. From figure 10 we see that the maximum velocity in the wave form due to the fundamental mode was $1{\cdot}42\,U$ and that the second mode produced, on the axis, a negative velocity of amplitude $0{\cdot}68U$. (The small oscillation between the two modes arises from the cessation of the obstacle motion.) Thus, taking into account the viscous effects, which have a greater influence on the second mode, it appears that the decomposition (5) gives a fairly good description of the motions.†

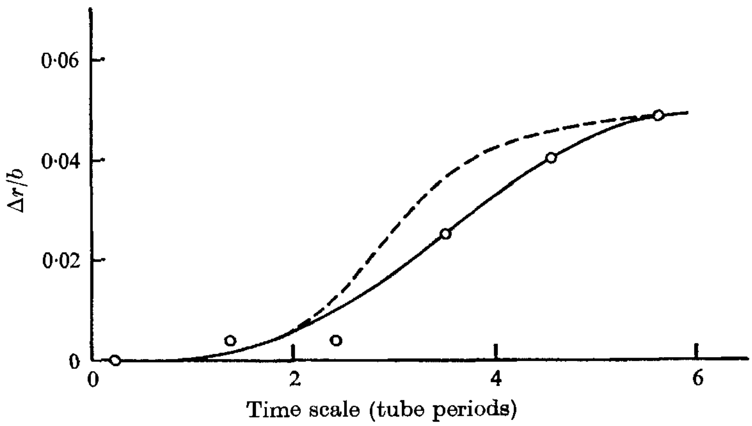

FIGURE 11. The radial displacements due to a wave form travelling along a tube. Measurements made at $r/b = 0{\cdot}522$, $U/\Omega b = 0{\cdot}054$, $a/b = 0{\cdot}71$. ———, (*a*) experiment; – – –, (*b*) deduced from the axial velocity distribution.

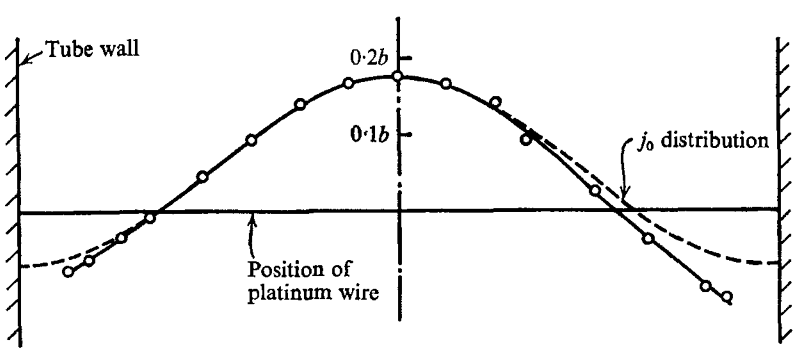

FIGURE 12. The distorted shape of a dye line; the shape is compared with a J_0 distribution of the same amplitude on the axis.

The radial displacements due to the fundamental mode have been measured. These measurements were made from a cylinder of dye produced by the wire stretched parallel to the axis; the results are shown in figure 11 where the diameter of this cylinder, at a fixed axial station, is given as a function of the time. The Rossby number, $U/\Omega b$, was $0{\cdot}054$ and a particle at a radius of $0{\cdot}55b$ was displaced by about $0{\cdot}05b$. The axial velocity distribution was also measured in this experi-

† To put these results into perspective the first two terms of the modal distribution have been calculated for slightly different velocity profiles. For a given x and t we find that: (*a*) if $u(r) = 1-\frac{3}{2}(r/b)$ then $\alpha_1 = 0{\cdot}880$; $\alpha_2 = -0{\cdot}112$; (*b*) if $u = 1-2(r/b)^2$ then $\alpha_1 = 1{\cdot}29$; $\alpha_2 = -0{\cdot}542$.

ment and was used in the way indicated below, to give an independent estimate of the radial displacements; these results are shown in figure 11 and there is good agreement between the two curves, especially if we take into account the fact that the direct measurement of the radial displacements is difficult.

The experiments indicate that the wave form due to the fundamental does not disperse much as it progresses along the tube and hence it is represented fairly well by

$$u = J_0(k_1 r)\,\phi(x-ct), \tag{6}†$$

where $c = 2\Omega/k_1$ and ϕ, the velocity on the axis within the advancing wave form, keeps a constant value at $x = (2\Omega/k_1)\,t$. The radial velocity follows from continuity and is

$$v = -\frac{J_1(k_1 r)}{k_1}\frac{\partial \phi}{\partial x}. \tag{7}$$

The radial displacement is now found by integrating (7) and is given by

$$\Delta r = J_1(k_1 r)\,(\phi_2 - \phi_1)/\Omega, \tag{8}$$

where $\phi_{1,2}$ are the velocities at the beginning and end of the time under consideration. This result was used to deduce the radial displacement from a measurement of the axial velocity distribution.

Knowing that the radial displacements are very much smaller than the axial displacements, we see, from (6), that a diametral line of particles will distort to the form $J_0(k_1 r)$. The measured shape of such a dye line is shown in figure 12 and is compared with the curve of $J_0(k_1 r)$ which has the same magnitude on the axis as the experimental curve: the agreement between the two curves is very good.

† Dr Kathleen Trustrum of the University of Sussex has pointed out to me that (6) is an exact solution of a 'long wave' approximation to the non-linear equations. With the 'long wave' approximation, $\partial v/\partial x$ is small compared with $\partial u/\partial r$ in the expression for the azimuthal component of the relative vorticity, and the equations for axisymmetric flow referred to axes rotating with angular velocity Ω about Ox are

$$-\frac{w^2}{r} - 2\Omega w = -\frac{1}{\rho}\frac{\partial P}{\partial r},$$

$$\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial r} + u\frac{\partial w}{\partial x} + \frac{vw}{r} + 2\Omega v = 0,$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x},$$

$$\frac{\partial}{\partial r}(rv) + \frac{\partial}{\partial x}(ru) = 0,$$

where (u, v, w) are the velocity components parallel to the (x, r, θ) directions. It is now easily demonstrated that

$$u = J_0(k_1 r)\,\phi(x-ct), \quad v = -\frac{J_1(k_1 r)}{k_1}\,\phi'(x-ct), \quad w = -J_1(k_1 r)\,\phi(x-ct),$$

$$\frac{P}{\rho} = cJ_0(k_1 r)\,\phi(x-ct) + (J_0^2 + J_1^2)\,\frac{\phi^2}{2},$$

where $c = 2\Omega/k_1$, is an exact solution of these equations. A similar solution holds for the two-dimensional flow of a Boussinesq fluid.

The wave motions as a function of the Rossby number

A series of measurements was made to determine the form of the motions at different Rossby numbers. The wave forms were generated by moving the body at a constant velocity along the axis a distance of 1·5 tube radii and then stopping the motion: the resulting wave form was observed at positions 25 and 40·5 tube radii from the initial position of the body. The body could be moved either towards or away from the observation points.

The theory of Lighthill (1967) indicates that, due to the Doppler effect, a given wave mode propagated ahead of the body is more powerfully excited than is the corresponding mode to the rear. Unfortunately it is difficult to observe this in a fluid of large radial extent, because of the continuous spectrum of wave modes, but in a long tube, where there is a discrete set of wave numbers, we can make observations on a single mode. Hence one of the purposes of this experiment is to see if the Doppler effect is observed in practice.

In each experiment the body was given the same displacement and was moved at the same speed; the Rossby number was varied by changing the angular velocity of the tube. Each pair of results (i.e. for the motions ahead of and behind the body) was obtained at approximately the same rotation rate Ω and therefore the effects of dispersion and viscous damping should be similar for both the wave forms of any pair of results.

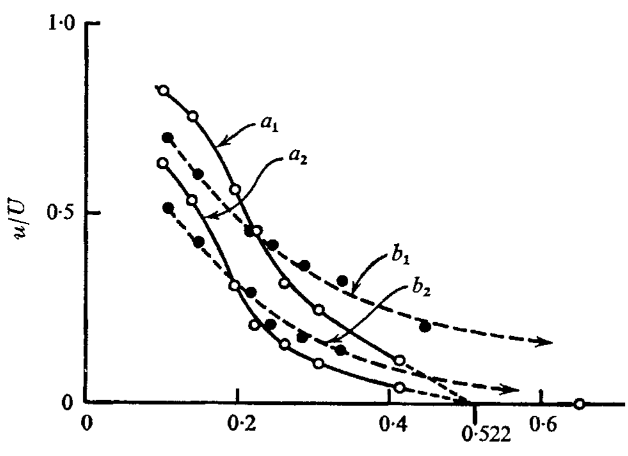

FIGURE 13. The motions, at various values of the Rossby number, generated by a body ($a/b = 0{\cdot}5$) moved along the axis of a tube. The flow was observed at positions $25b$ (subscript 1) and $40{\cdot}5b$ (subscript 2) from the body. ———, (a) forward motions; – – – –, (b) motions at the rear.

The results of the experiments are shown in figure 13. At the lower Rossby numbers (e.g. at $U/\Omega b \sim 0{\cdot}1$) the velocity ahead of the body was significantly greater than that behind, as indicated by the theory of Lighthill (1967). However, contrary to the theory, the waves at the higher Rossby numbers were not as powerfully excited ahead of the body as those behind, but this is probably because the group velocity of the waves only slightly exceeds the velocity of the body. As a result the motion of the body does not last for a sufficient period of time to excite the flow ahead to near its ultimate steady state.

The results also indicate that the particle velocity in the forward travelling wave form tends to zero as the velocity of the body approaches the maximum group velocity in the tube, $0{\cdot}522\Omega b$. An experiment made at a Rossby number of 0·65 showed no motion at all ahead of the body, whereas significant motions behind the body were observed at this Rossby number. Thus it appears that no (finite) disturbances propagate faster than the group velocity, suggesting that waves of finite amplitude and permanent form do not exist in this situation, in agreement with the analysis of Benjamin (1967).

The maximum group velocity

It is fairly difficult in an experiment to make an exact measurement of the group velocity. Benjamin & Barnard (1964) have made a rough measurement of the group velocity by determining the speed of propagation of an axial velocity in the Taylor column of half the velocity of the obstacle. Their estimate of the maximum group velocity was $0{\cdot}48\Omega b$.

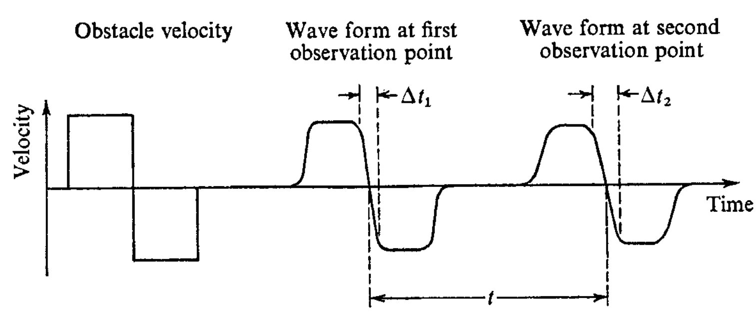

FIGURE 14. Illustration of the method used to measure the group velocity.

Attempts have been made during this investigation to measure the group velocity by determining the speed of propagation of a discontinuity in the wave form: the idea is illustrated in figure 14, from which we see that the discontinuity was produced by suddenly reversing the direction of motion of the body. As the wave travels along the tube the discontinuity becomes less distinct, due to dispersion, thereby introducing an uncertainty to the measurement, indicated approximately in figure 14 by the Δt's. Accordingly it was decided to characterize the discontinuity by that part of the wave form corresponding to zero particle velocity. Because the wave form dispersed by only a small amount between the two observation points this characterization should be fairly accurate. Some measurements of the group velocity (C_g), are shown in table 1: the mean value of these results gives an estimate of $0{\cdot}512\Omega b$ (with a standard deviation of $0{\cdot}092\Omega b$) which is very close to the theoretical value of $0{\cdot}522\Omega b$.

Experiment	A	B	C	D	E	F	G
$U/\Omega b$	0·029	0·067	0·050	0·054	0·049	0·045	0·045
$C_g/\Omega b$	0·526	0·510	0·514	0·521	0·509	0·509	0·496

Notes: (i) The body was moved towards the observation points and then away from them for all the experiments except F, where the opposite is true. (ii) Experiment G was made with an annular body, i.e. one which blocked the area of the tube in the annulus between the radius a and the wall of the tube. (iii) Experiment E was made in a liquid with a kinematic viscosity 3·09 times greater than that of water at 20 °C.

TABLE 1

5. General comments

Experiments made at very low Rossby numbers (Taylor 1922; Long 1953; Maxworthy 1968) show that fluid is pushed ahead of a body moving along the axis. Taylor (1922) and (Long 1953) also observed the flow at Rossby numbers around 0·2–0·4 and both found that, above some critical value of the Rossby number, the body no longer pushes a column of fluid ahead of itself, but their estimates for the 'critical' Rossby number differ widely—about $1/\pi$ in Taylor's experiments and 0·23 in Long's. No adequate theoretical calculation has yet been made to estimate this 'critical' Rossby number.

The present experiments have been made over a much wider range of Rossby numbers than covered by the previous workers and the results suggest that, even at very high Rossby numbers, there is a significant influence far ahead of the body. The results of the experiments made in the long tube are in very good agreement with the theoretical models (cf. Lighthill 1967; Benjamin & Barnard 1964): the form of the disturbances agrees very closely with a normal-mode theory and an approximate measurement of the maximum group velocity is near the theoretical value; in addition, the results indicate that no motions propagate ahead of the body when its velocity exceeds the maximum group velocity of the waves. An interesting feature of these experiments is that the body may generate disturbances in which the particle velocities on the axis exceed the velocity of the body.

In a fluid of large radial extent the forward influence of a sphere has been measured as a function of the Rossby number: the results are compared (cf. figure 6) with a fairly crude theoretical estimate of this function and the two are in very good agreement considering the approximations made in the calculation. Thus in view of the good general agreement of the experimental results with the predictions made from the low Rossby number theories it is felt that the motion of the sphere generates the complete spectrum of wave modes, implying (see Lighthill 1967) that there is a forward influence at all Rossby numbers, a contention which is supported by the theoretical work of Stewartson (1958, 1968), and in particular by the work of Benjamin (1969).

If these contentions are correct, it then appears that the interpretations of the low Rossby number theories used to make the theoretical estimates shown in figure 6 may give a fairly good indication of the flow field ahead of the body at all Rossby numbers. Thus, following Greenspan's (1968) calculation, we have roughly sketched in figure 15 the form we expect the flow field to take. At large Rossby numbers no fluid is blocked by the sphere, although a finite influence extends far upstream, and at very large Rossby numbers the streamline pattern begins to take on the appearance of the familiar potential-flow pattern. When

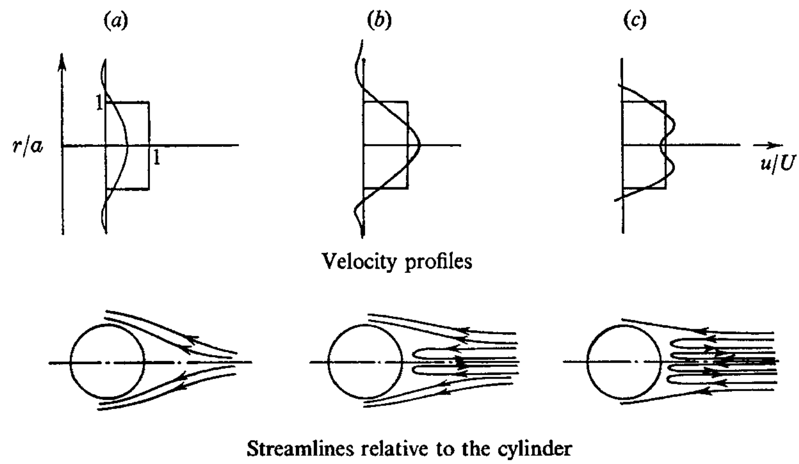

FIGURE 15. A sketch of the form of the conjectured flow patterns at various Rossby numbers. (a) $U/\Omega a > 0{\cdot}68$; (b) $0{\cdot}33 < U/\Omega a < 0{\cdot}68$; (c) the first pattern for $U/\Omega a < 0{\cdot}33$.

the Rossby number drops below about 0·68, according to the interpretation of Greenspan's analysis, the particle velocity on the axis exceeds the velocity of the body and the necessary condition for the blocking effect is satisfied; thus the possibility arises of closed streamlines ahead of the body, as indicated in figure 15 (*b*). At lower Rossby numbers the flow pattern becomes more complicated (cf. figure 15(*c*)).

These ideas are largely conjectural (and they neglect the possibility that these flows may not be stable), but if the motions do develop in the way described then the salient features of the flows (such as particle velocities well in excess of the velocity of the body) should readily be observed in experiments.† We recall that, in the long cylindrical tube, particle velocities in excess of the velocity of the body have been observed.

Thus, from the results for the motions ahead of a sphere as a function of the Rossby number (cf. figure 6), and from considerations of the kind just described it is felt that Greenspan's conjecture on the blocking phenomenon is fairly accurate, so that an ever-lengthening column of fluid is trapped in front of the sphere at all values of the Rossby number less than a critical value which is about 0·7.

† The apparatus used for these experiments unfortunately could not be used at the lower Rossby numbers.

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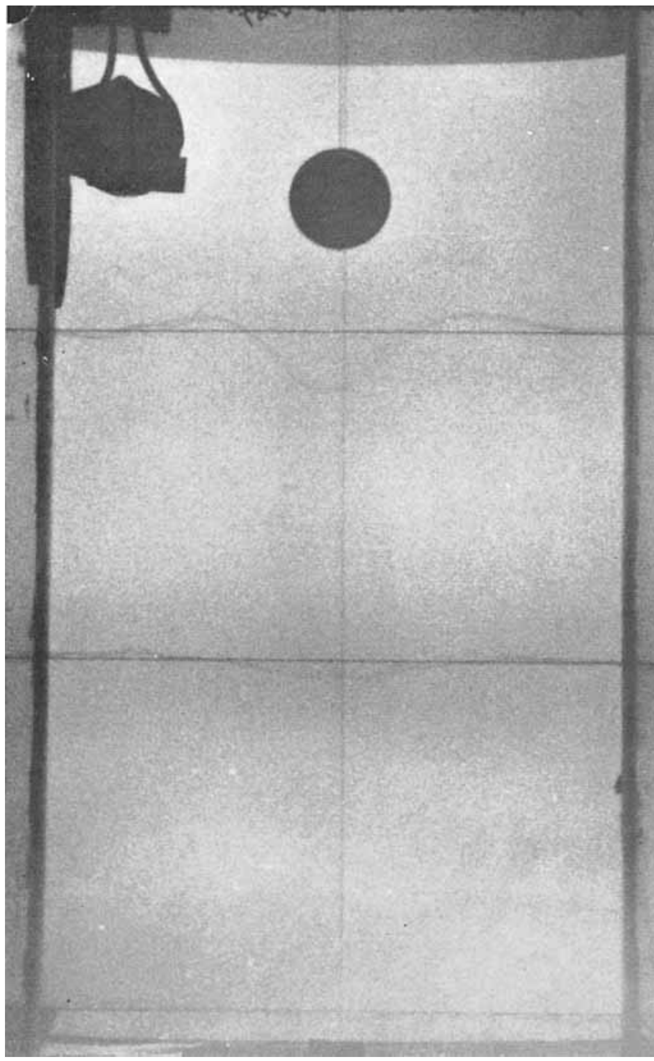

FIGURE 2. The distortion of a dye line due to the motion of a sphere towards it. The original position of the dye lines coincided with the wires that stretch transversely across the tube. The sphere has travelled a distance $12a$ along the axis at a Rossby number $(U/\Omega a)$ of 0·834. The wire stretched in the axial direction is attached to the wall of the container.

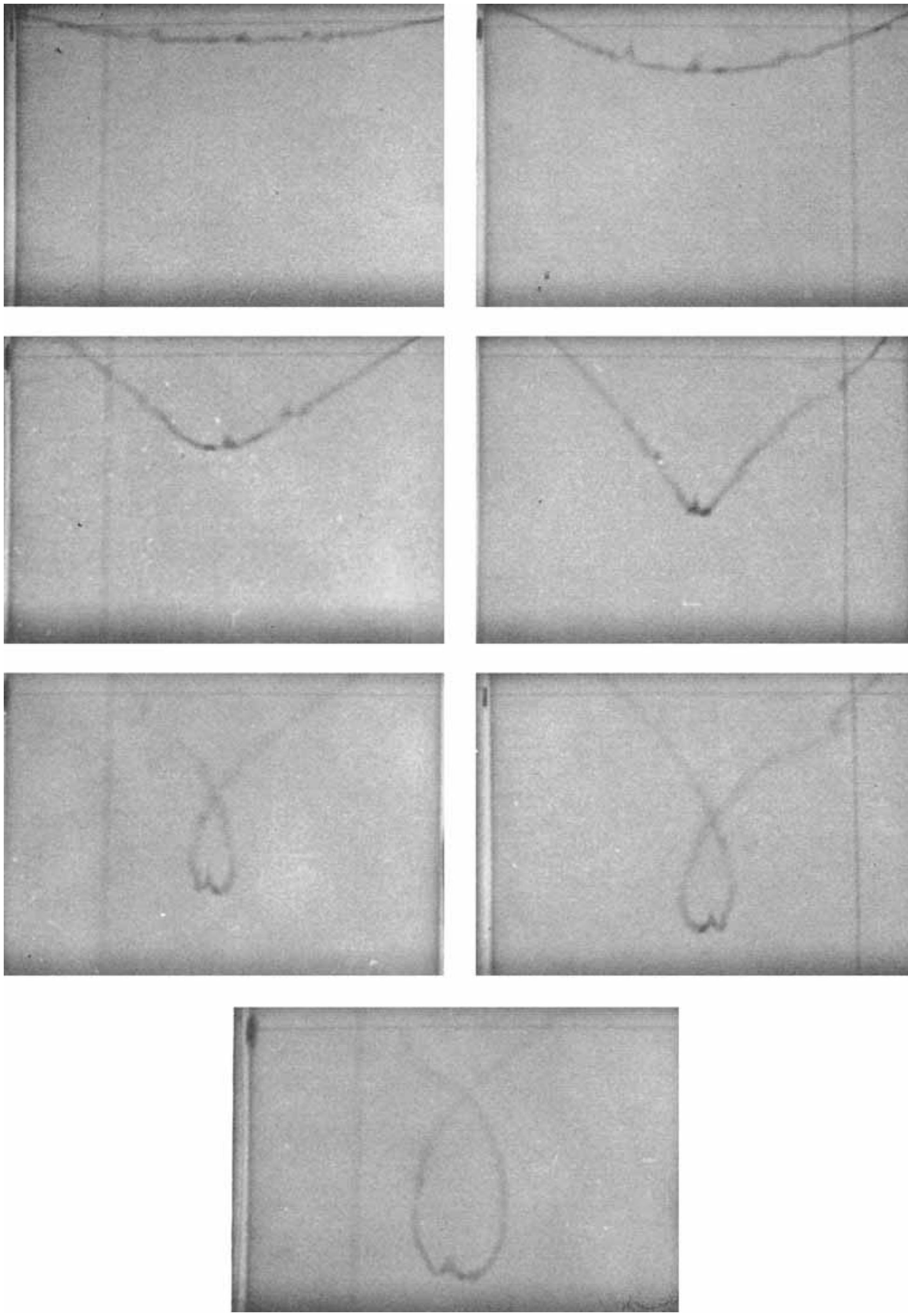

FIGURE 7. A sequence of photographs showing, at successive half-periods of rotation, the passage of a wave along a tube. The observation point is $25b$ from the obstacle; $U/\Omega b = 0{\cdot}055$.